A Tractable Model for Turbulence- and Misalignment-Induced Fading in Optical Wireless Systems

Harilaos G. Sandalidis, Nestor D. Chatzidiamantis, and George K. Karagiannidis, *Fellow, IEEE*

***Abstract*— Composite models for the combined effect of both scintillation- and misalignment-induced fading in terrestrial optical wireless systems are often expressed in terms of higher order special functions, making, thus, the performance evaluation of such links quite complicated. In order to overcome this problem, we propose the mixture Gamma distribution as an accurate approximation of the turbulence effect. Then, a tractable mixture distribution is deduced, for the statistical description of the composite scintillation/pointing error effects. Some statistical metrics of the new model are also derived in closed form, which provides insight information for the system design.**

***Index Terms*— Terrestrial optical wireless (OW) communications, mixture distribution, turbulence, pointing errors.**

I. Introduction

The derivation of a composite model to describe the combined effect of both scintillation and misalignment induced fading in terrestrial optical wireless (OW) systems has been gaining an increased research interest [1]. In the pioneering work [2], the authors considered a versatile pointing error model as well as lognormal and K distributions to model the turbulence but they did not derived the composite probability density functions (PDFs) in closed form. Furthermore, in [3], an analytical expression assuming Gamma-Gamma turbulence was presented, while the Málaga model was used in [4]. However, the produced composite PDFs are extremely difficult to be treated, as they are expressed in terms of higher-order special functions (e.g., Meijer-G function).

In this letter, we propose an alternative approach, by adopting the mixture Gamma (MG) distribution to model the turbulence effect. The MG PDF is a linearly weighted sum of Gamma PDFs and has been suggested in [5] to model the signal-to-noise ratio of composite shadowing/fading in RF wireless systems. Two of the commonly used turbulence models, i.e., the Gamma-Gamma and the Málaga distributions are represented in the form of the MG model. The accuracy of the MG model is examined by computing the Kullback-Leibler (KL) divergence. Then, another mixture distribution for the composite scintillation/pointing error channel using the jitter model in [2], is deduced. The fundamental statistical metrics are then derived in closed form.

Manuscript received May 21, 2016; accepted June 18, 2016. Date of publication June 23, 2016; date of current version September 8, 2016. The associate editor coordinating the review of this letter and approving it for publication was J. Cheng.

H. G. Sandalidis is with the Department of Computer Science and Biomedical Informatics, University of Thessaly, Lamia 35 131, Greece (e-mail: sandalidis@dib.uth.gr).

N. D. Chatzidiamantis and G. K. Karagiannidis are with the Department of Electrical and Computer Engineering, Aristotle University of Thessaloniki, Thessaloniki 54 124, Greece (e-mail: nestoras@auth.gr; geokarag@auth.gr).

Digital Object Identifier 10.1109/LCOMM.2016.2584612

II. The Mixture Gamma Turbulence Model

A. The Distribution

The PDF of the irradiance due to scintillation effect, I_α, is approximated by the MG model according to [5], i.e.,

$$f_{I_\alpha}(I_\alpha) = \sum_{i=1}^{N} w_i f_i(I_\alpha) = \sum_{i=1}^{N} a_i I_a^{b_i-1} e^{-\zeta_i I_a}, \quad I_a > 0, \tag{1}$$

where N is the number of terms, $f_i(x) = \frac{\zeta_i^{b_i} x^{b_i-1}\exp(-\zeta_i x)}{\Gamma(b_i)}$ is the PDF of a Gamma distribution with parameters a_i, b_i, and ζ_i, $w_i = \frac{a_i \Gamma(b_i)}{\zeta_i^{b_i}}$ with $\sum_{i=1}^{N} w_i = 1$, and $\Gamma(\cdot)$ is the Gamma function [6, eq. (8.310.1)].

According to [5], the cumulative distribution function (CDF) of I_α is given as

$$F_{I_\alpha}(I_\alpha) = \sum_{i=1}^{N} a_i \zeta_i^{-b_i} \gamma(b_i, \zeta_i I_\alpha), \tag{2}$$

where $\gamma(\cdot,\cdot)$ is the lower incomplete Gamma function defined in [6, eq. (8.350.1)].

B. Approximation of Gamma-Gamma Distribution

The Gamma-Gamma PDF is expressed as

$$f_{I_\alpha}(I_\alpha) = \frac{2(\alpha\beta)^{(\alpha+\beta)/2}}{\Gamma(\alpha)\Gamma(\beta) I_\alpha}\left(\frac{I_\alpha}{\overline{I}_\alpha}\right)^{\frac{\alpha+\beta}{2}} K_{\alpha-\beta}\left(2\sqrt{\alpha\beta\frac{I_\alpha}{\overline{I}_\alpha}}\right), \quad I_a > 0, \tag{3}$$

where $\overline{I}_\alpha$ is the mean irradiance, $K_\nu(\cdot)$ is the νth-order modified Bessel function of the second kind [6, eq. (8.432.1)], and α, β are the effective number of small-scale and large-scale eddies of the scattering environment, respectively.

Using the integral representation of the function $K_\nu(\cdot)$ [6, eq. (8.432.6)] and after some mathematical manipulations, (3) can be rewritten as [5].

$$f_{I_\alpha}(I_\alpha) = \frac{(\alpha\beta)^\alpha I_\alpha^{\alpha-1}}{\overline{I}_\alpha^\alpha \Gamma(\alpha)\Gamma(\beta)} \int_0^\infty e^{-t} t^{-\alpha+\beta-1} e^{-\frac{\alpha\beta I_\alpha}{\overline{I}_\alpha t}} dt = \frac{(\alpha\beta)^\alpha I_\alpha^{\alpha-1}}{\overline{I}_\alpha^\alpha \Gamma(\alpha)\Gamma(\beta)} \int_0^\infty e^{-t} g(t)\, dt, \tag{4}$$

where $g(t) = t^{-\alpha+\beta-1}\exp\left(-\frac{\alpha\beta I_\alpha}{\overline{I}_\alpha t}\right)$. The integral in (4) can be approximated using the Gaussian-Laguerre quadrature method, in terms of $\sum_{i=1}^{N} w_i g(t_i)$, where w_i and t_i are the weight factors and the abscissas, respectively [7, Table 25.9].

By matching the two PDFs (1) and (4), the parameters of the MG distribution are extracted as[1]

$$a_i = \frac{\theta_i}{\sum_{j=1}^{N} \theta_j \Gamma\left(b_j\right) \zeta_j^{-b_j}}, \quad b_i = \alpha,$$

$$\zeta_i = \frac{\alpha\beta}{\overline{I}_\alpha t_i}, \quad \theta_i = \frac{(\alpha\beta)^\alpha w_i t_i^{-\alpha+\beta-1}}{\overline{I}_\alpha^\alpha \Gamma(\alpha)\Gamma(\beta)}. \tag{5}$$

C. Approximation of Málaga Distribution

The PDF of the Málaga distribution is given by [4]

$$f_{I_\alpha}\left(I_\alpha\right) = A \sum_{k'=1}^{\beta'} \alpha'_{k'} I_\alpha^{\frac{\alpha'+k'}{2}-1} K_{\alpha-k'}\left(2\sqrt{\frac{\alpha'\beta' I_\alpha}{\gamma'\beta'+\Omega'}}\right), \quad I_\alpha > 0, \tag{6}$$

where

$$A \triangleq \frac{2\alpha'^{\frac{\alpha'}{2}}}{\gamma'^{1+\frac{\alpha'}{2}}\Gamma\left(\alpha'\right)}\left(\frac{\gamma'\beta'}{\gamma'\beta'+\Omega'}\right)^{\beta'+\frac{\alpha'}{2}},$$

$$\alpha'_{k'} \triangleq \binom{\beta'-1}{k'-1}\frac{\left(\gamma'\beta'+\Omega'\right)^{1-\frac{k'}{2}}}{\left(k'-1\right)!}\left(\frac{\Omega'}{\gamma'}\right)^{k'-1}\left(\frac{\alpha'}{\beta'}\right)^{\frac{k'}{2}},$$

$$\Omega' \triangleq \Omega + 2\rho b_0 + 2\sqrt{2\rho b_0 \Omega}\cos\left(\varphi_A - \varphi_B\right). \tag{7}$$

In the above, Ω is the average power of the LOS component, $2b_0$ is the average power of the total scatter components, β' is a natural number representing the amount of turbulence, $\gamma' = 2b_0(1-\rho)$, α' is a positive parameter depending on the effective number of large-scale cells of the scattering process, ρ $(0 < \rho < 1)$ is the amount of scattering power coupled to the LOS component, and φ_A, φ_B are the deterministic phases of the LOS and the coupled-to-LOS components. Note that when $\rho = 1$ and $\Omega' = 1$, the Málaga PDF reduces to the Gamma-Gamma distribution [4].

Using [6, eq. (8.432.6)] and after some algebra, (6) takes the form of (8), as shown at the bottom of this page. Then, the parameters of the MG PDF can be readily derived by comparing (1) and (8) as

$$a_i = \frac{\theta_i}{\sum_{j=1}^{N} \theta_j \Gamma\left(b_j\right) \zeta_j^{-b_j}}, \quad b_i = \alpha',$$

$$\zeta_i = \frac{\alpha'\beta'}{\left(\gamma'\beta'+\Omega'\right) t_i}, \quad \theta_i = \frac{A}{2} w_i \sum_{k'=1}^{\beta'} \alpha'_{k'} \left(\frac{\alpha'\beta'}{\gamma'\beta'+\Omega'}\right)^{\frac{\alpha'-k'}{2}} t^{k'-\alpha'-1}. \tag{9}$$

[1] Alternatively, since $K_{\alpha-\beta}(\cdot) = K_{\beta-\alpha}(\cdot)$, the use of [6, eq. (8.432.6)] also leads to the following parameters: $b_i = \beta$ and $\theta_i = \frac{(\alpha\beta)^\beta w_i t_i^{-\beta+\alpha-1}}{\overline{I}_\alpha^\beta \Gamma(\alpha)\Gamma(\beta)}$.

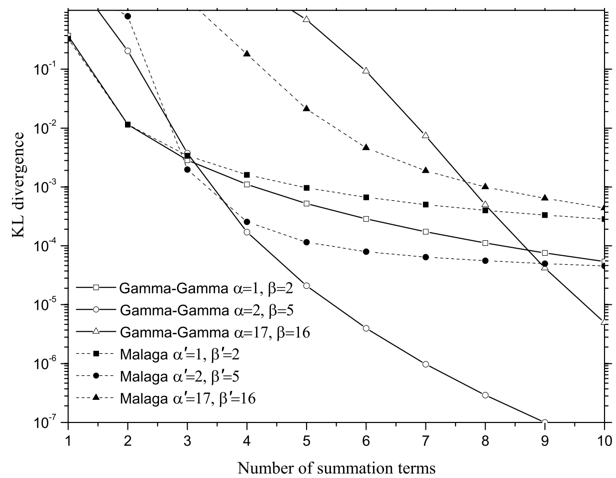

Fig. 1. KL divergence vs. N for several exact and approximated PDFs.

D. Accuracy of the Approximation

Very recently, models of the turbulence induced fading with PDFs based on infinite series expansions, appeared in the technical literature, e.g., [8], [9]. These PDFs were mainly proposed in order to simplify the derivation of metrics such as the bit-error-rate. However, to keep the PDF property, they require an infinite number of summations, since when this number is finite, the integral over the entire space is not equal to one. The difference in our approach lies in the fact that the PDF property is still preserved for any finite number of terms, since the integral over the entire space is always one due to the formulation discussed in II.A.

Roughly speaking, the accuracy of the approximation of a given distribution with the MG model depends on the number of components, N. An appropriate number can be selected as the minimum value such that the KL divergence between the exact and the MG distribution is below a given threshold. The D_{KL} divergence is defined as

$$D_{KL} \triangleq \int_0^\infty f_{I_\alpha}\left(I_\alpha\right) \ln \frac{f_{I_\alpha}\left(I_\alpha\right)}{\widehat{f}_{I_\alpha}\left(I_\alpha\right)} dI_\alpha, \tag{10}$$

where $f_{I_\alpha}(I_\alpha)$ is the exact and $\widehat{f}_{I_\alpha}(I_\alpha)$ the approximated (MG) PDF. The approximation can be as accurate as we need by simply increasing the number of terms.

Figure 1 demonstrates the KL divergence for some typical values of the Gamma-Gamma and Málaga distributions which characterize weak and strong turbulence conditions. For the first distribution we consider that $\overline{I}_\alpha = 1$, whereas for the last one, $\rho = 0.9$ and $\Omega + 2b_0 = 1$. In all cases, the KL divergence is set below 10^{-3}, when the number of components $N \geq 10$. Therefore, it can be assured that the MG distribution is an accurate approximation of the above distributions, even for a small number of terms, N. This is also clear in Figs. 2 and 3

$$f_{I_\alpha}\left(I_\alpha\right) = \frac{1}{2} A \sum_{k=1}^{\beta'} \alpha'_{k'} I_\alpha^{\alpha'-1} \left(\frac{\alpha'\beta'}{\gamma'\beta'+\Omega'}\right)^{\frac{\alpha'-k'}{2}} \int_0^\infty e^{-t} t^{k'-\alpha'-1} \exp\left(-\frac{\alpha'\beta' I_\alpha}{\left(\gamma'\beta'+\Omega'\right) t}\right) dt$$

$$= I_\alpha^{\alpha'-1} \int_0^\infty e^{-t} \frac{1}{2} A \sum_{k=1}^{\beta'} \alpha'_{k'} \left(\frac{\alpha'\beta'}{\gamma'\beta'+\Omega'}\right)^{\frac{\alpha'-k}{2}} t^{k'-\alpha'-1} \exp\left(-\frac{\alpha'\beta' I_\alpha}{\left(\gamma'\beta'+\Omega'\right) t}\right) dt. \tag{8}$$

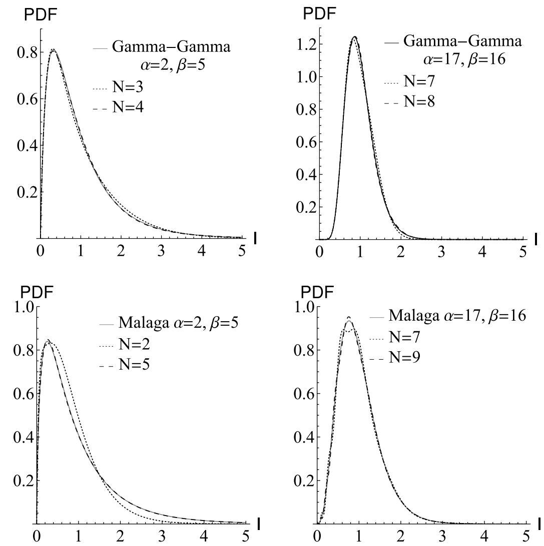

Fig. 2. Plots for several exact and approximated PDFs.

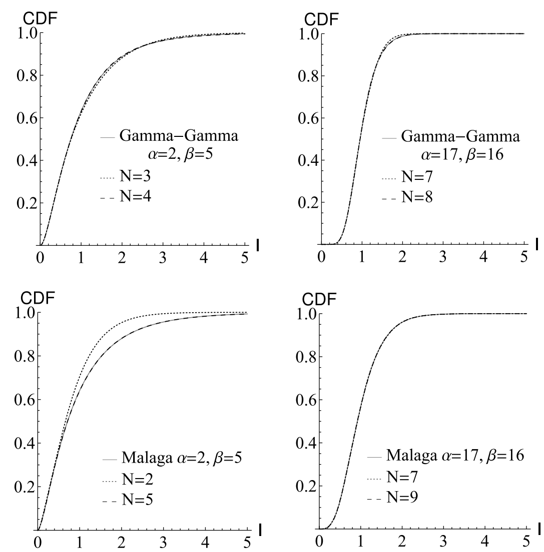

Fig. 3. Plots for several exact and approximated CDFs.

where the PDF and CDF for some exact and approximated distributions are depicted.

In general, the number of terms depends on the parameter values characterizing the exact PDF (e.g. parameters α and β for the gamma-gamma distribution). Therefore, in the case where these parameters vary with time, there is a need to continuously re-estimate N, in order to achieve convergence. An adaptive algorithm may be employed for this task. However, a selection of a large value of N, let's say 20, usually implies sufficient convergence for almost all parameter values.

III. Misalignment Induced Fading Model

The PDF of the irradiance due to pointing error effects, I_p, can be expressed as [2]

$$f_{I_p}\left(I_p\right) = \frac{\gamma^2}{A_0^{\gamma^2}} I_p^{\gamma^2-1}, \quad 0 \leq I_p \leq A_0. \tag{11}$$

In the above equation, $\gamma = w_{z_{eq}}/2\sigma_s$ is the ratio between the equivalent beam radius at the receiver and the pointing error displacement standard deviation (jitter) at the receiver, $w_{z_{eq}}^2 = w_z^2 \sqrt{\pi}\,\mathrm{erf}\left(\upsilon\right)/2\upsilon \exp\left(-\upsilon^2\right)$, $\upsilon = \sqrt{\pi} r/\sqrt{2} w_z$, $A_0 = [\mathrm{erf}\left(\upsilon\right)]^2$, and $\mathrm{erf}(\cdot)$ is the error function [6, eq. (8.250.1)].

It has to be noted here that this model assumes a Gaussian spatial intensity profile of beam waist radius, w_z, on the receiver plane at distance z from the transmitter and a circular aperture of radius r. Moreover, independent identical Gaussian distributions for the elevation and the horizontal displacement (sway) both of variance σ_s^2 are considered.

IV. Mixture Composite Irradiance Model

A. PDF

The composite PDF of irradiance, $I = I_\alpha I_p$, is derived using the formula [2]

$$f_I\left(I\right) = \int f_{I|I_a}\left(I|I_a\right) f_{I_a}\left(I_a\right) d\,I_a, \tag{12}$$

where $f_{I|I_a}\left(I|I_a\right)$ is the conditional probability given I_a state and is expressed as

$$\begin{aligned} f_{I|I_a}\left(I|I_a\right) &= \frac{1}{I_a} f_{I_p}\left(\frac{I}{I_a}\right) \\ &= \frac{\gamma^2}{A_0{}^{\gamma^2} I_a}\left(\frac{I}{I_a}\right)^{\gamma^2-1}, \quad 0 \leq I \leq A_0 I_a. \end{aligned} \tag{13}$$

By placing (13) and (1) into (12) and after some algebra, we get

$$f_I\left(I\right) = \frac{\gamma^2}{A_0{}^{\gamma^2}} I^{\gamma^2-1} \sum_{i=1}^{N} \int_{I/A_0}^{\infty} a_i I_a^{b_i-\gamma^2-1} \exp(-\zeta_i I_a) d\,I_a \tag{14}$$

and an analytical result is derived using [6, eq. (3.381.3)] as

$$f_I\left(I\right) = \frac{\gamma^2}{A_0{}^{\gamma^2}} I^{\gamma^2-1} \sum_{i=1}^{N} a_i \zeta_i^{\gamma^2-b_i} \Gamma\left(b_i - \gamma^2, \frac{\zeta_i}{A_0} I\right), \quad I > 0, \tag{15}$$

where $\Gamma\left(\cdot,\cdot\right)$ is the upper incomplete Gamma function defined in [6, eq. (8.350.2)].[2]

B. CDF

The CDF of I results from (15) as

$$\begin{aligned} F_I\left(I\right) &\triangleq \int_0^I f_I(I) dI \\ &= \int_0^I \frac{\gamma^2}{A_0{}^{\gamma^2}} I^{\gamma^2-1} \sum_{i=1}^{N} a_i \zeta_i^{\gamma^2-b_i} \Gamma\left(b_i - \gamma^2, \frac{\zeta_i}{A_0} I\right), \end{aligned} \tag{16}$$

which reduces after some manipulations to

$$F_I\left(I\right) = \gamma^2 \sum_{i=1}^{N} a_i \zeta_i^{-b_i} \int_0^{\frac{\zeta_i I}{A_0}} I^{\gamma^2-1} \Gamma\left(b_i - \gamma^2, I\right). \tag{17}$$

[2]Note that (15) is much simpler and more tractable than the composite models assuming Gamma-Gamma or Málaga distributions, since the latter are given in terms of higher-order special functions, such as the Meijer's G-function (see, e.g., [3], [4]).

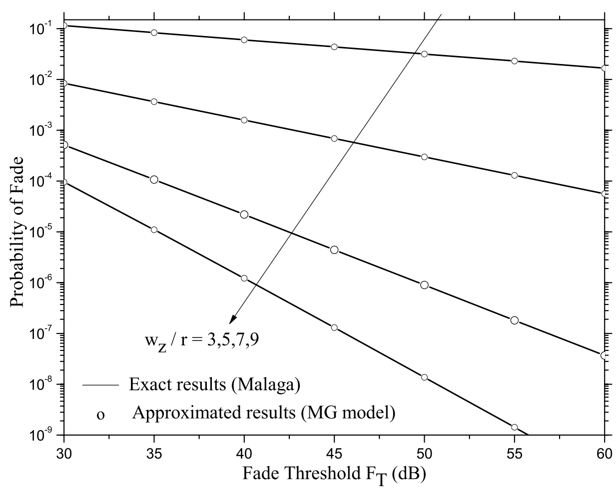

Fig. 4. Probability of fade vs FT.

Then a closed-form expression is extracted using [6, eq. (8.356.3)] and [10, eq. (06.06.21.0002.01)] as

$$F_I(I) = \sum_{i=1}^{N} a_i \zeta_i^{-b_i} \left[\left(\frac{\zeta_i I}{A_0} \right)^{\gamma^2} \Gamma\left(b_i - \gamma^2, \frac{\zeta_i I}{A_0} \right) + \gamma\left(b_i, \frac{\zeta_i I}{A_0} \right) \right]. \quad (18)$$

C. Moments

The k-th order moment is defined as

$$\mu_I(k) \triangleq \int_0^\infty I^k f_I(I) dI = \gamma^2 A_0^k \sum_{i=1}^{N} a_i \zeta_i^{-k-b_i} \int_0^\infty I^{k+\gamma^2-1} \Gamma\left(b_i - \gamma^2, I \right) dI, \quad (19)$$

where an analytical solution is derived utilizing [7, eq. (6.5.37)] as

$$\mu_I(k) = \frac{\gamma^2 A_0^k}{k+\gamma^2} \sum_{i=1}^{N} a_i \zeta_i^{-(k+b_i)} \Gamma(k+b_i). \quad (20)$$

D. Scintillation Index (SI)

The SI occurs as [11]

$$SI \triangleq \frac{E\left[I^2\right] - E^2[I]}{E^2[I]} = \frac{\left(1+\gamma^2\right)^2 \sum_{i=1}^{N} a_i \zeta_i^{-(2+b_i)} \Gamma(2+b_i)}{\gamma^2\left(2+\gamma^2\right) \left(\sum_{i=1}^{N} a_i \zeta_i^{-(1+b_i)} \Gamma(1+b_i) \right)^2} - 1, \quad (21)$$

where $E[\cdot]$ denotes the expected value of the enclosed.

E. Moment Generating Function (MGF)

The MGF is obtained as

$$M_I(s) \triangleq \int_0^\infty e^{-sI} f_I(I) dI = \frac{\gamma^2}{A_0{}^{\gamma^2}} \sum_{i=1}^{N} a_i \zeta_i^{\gamma^2 - b_i} \times \int_0^\infty I^{\gamma^2-1} \exp(-sI)\, \Gamma\left(b_i - \gamma^2, \frac{\zeta_i}{A_0} I \right) dI, \quad (22)$$

which can be solved using [6, eq. (6.455.1)] as

$$M_I(s) = \sum_{i=1}^{N} a_i \frac{\Gamma(b_i)}{(\zeta_i + A_0 s)^{b_i}} F\left(1, b_i; \gamma^2+1; \frac{A_0 s}{\zeta_i + A_0 s} \right), \quad (23)$$

where $F(\cdot, \cdot; \cdot; \cdot)$ is the hypergeometric series [6, eq. (9.100)].

F. Probability of Fade

The probability of fade is derived from (18) as $F_I(I_T)$ where I_T is the threshold level of the irradiance [11]. It can be also expressed in terms of the fade threshold parameter, F_T, as [11]

$$F_T \triangleq 10 \log_{10} \left(\frac{\overline{I}}{I_T} \right) \quad [dB], \quad (24)$$

where $\overline{I} = \mu_I(1)$. As an example, Fig. 4 illustrates the probability of fade in terms of F_T for several values of normalized beamwidth. N is selected as the minimum value to satisfy $D_{kl} \leq 10^{-4}$, i.e., $N = 15$. The following parameter values are used: $\Omega + 2b_0 = 1$, $\acute{\alpha} = 2$, $\acute{\beta} = 5$, $\rho = 0.7$, $\varphi_A - \varphi_B = \frac{\pi}{2}$, and $\sigma_s/r = 3$. The derived results reveal that the probability of fade increases for small values of w_z/r.

V. Conclusions

A novel mixture distribution to describe the combined effect of turbulence and misalignment-induced fading was derived in this letter. The new model was deduced by using the MG distribution to describe turbulence. The simplicity and tractability are some of the benefits of our approach.

References

[1] X. Song, F. Yang, and J. Cheng, "Subcarrier intensity modulated optical wireless communications in atmospheric turbulence with pointing errors," *IEEE/OSA J. Opt. Commun. Netw.*, vol. 5, no. 4, pp. 349–358, Apr. 2013.

[2] A. A. Farid and S. Hranilovic, "Outage capacity optimization for free-space optical links with pointing errors," *J. Lightw. Technol.*, vol. 25, no. 7, pp. 1702–1710, Jul. 2007.

[3] H. G. Sandalidis, T. A. Tsiftsis, and G. K. Karagiannidis, "Optical wireless communications with heterodyne detection over turbulence channels with pointing errors," *J. Lightw. Technol.*, vol. 27, no. 20, pp. 4440–4445, Oct. 15, 2009.

[4] A. Jurado-Navas, J. M. Garrido-Balsells, J. F. Paris, M. Castillo-Vázquez, and A. Puerta-Notario, "Impact of pointing errors on the performance of generalized atmospheric optical channels," *Opt. Exp.*, vol. 20, no. 11, pp. 12550–12562, 2012.

[5] S. Atapattu, C. Tellambura, and H. Jiang, "A mixture gamma distribution to model the SNR of wireless channels," *IEEE Trans. Wireless Commun.*, vol. 10, no. 12, pp. 4193–4203, Dec. 2011.

[6] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*. 7th ed. New York, NY, USA: Academic, 2008.

[7] M. Abramowitz and I. A. Stegun, Eds., *Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables*, 9th ed. New York, NY, USA: Dover, 1972.

[8] M. R. Bhatnagar and Z. Ghassemlooy, "Performance analysis of gamma–gamma fading FSO MIMO links with pointing errors," *J. Lightw. Technol.*, vol. 34, no. 9, pp. 2158–2169, May 1, 2016.

[9] M. R. Bhatnagar, "A one bit feedback based beamforming scheme for FSO MISO system over gamma-gamma fading," *IEEE Trans. Commun.*, vol. 63, no. 4, pp. 1306–1318, Apr. 2015.

[10] *The Wolfram Functions Site*. (2016). [Online]. Available: http://functions.wolfram.com

[11] L. C. Andrews and R. L. Phillips, *Laser Beam Propagation Through Random Media*, 2nd ed. Bellingham, WA, USA: SPIE, 2005.